Binomial Theorem

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Pascal's Triangle

- Some common expansions are given as
- $(a + b)^0 = 1$
- $(a + b)^1 = a + b$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a + b)^4 = (a + b)^2 (a + b)^2 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- The index of each expansion and the coefficients of the terms in the expansions are different. They however, share a relationship, which is given by ***Pascal's Triangle***, which is shown below.

Index	Coefficients
0	${}^0C_0$ (= 1)
1	${}^1C_0$ (= 1) ${}^1C_1$ (= 1)
2	${}^2C_0$ (=1) ${}^2C_1$ (=2) ${}^2C_2$ (= 1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (= 1) ${}^4C_1$ (= 4) ${}^4C_2$ (= 6) ${}^4C_3$ (= 4) ${}^4C_4$ (= 1)
5	${}^5C_0$ (= 1) ${}^5C_1$ (= 5) ${}^5C_2$ (= 10) ${}^5C_3$ (= 10) ${}^5C_4$ (= 5) ${}^5C_5$ (= 1)

- Pascal's triangle can be continued endlessly and can be used for writing the coefficients of the terms occurring in the expansion of $(a + b)^n$.
- For example, look at the row corresponding to index 5. It can be used for expanding $(a + b)^5$ as $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Binomial Theorem

- Binomial theorem is used for expanding the expressions of the type $(a + b)^n$, where n can be a very large positive integer.

- The binomial theorem states that the expansion of a binomial for any positive integer n is given by $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots\ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$

  - **The binomial theorem can also be stated as** $(a + b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$

  - **The coefficients** ${}^nC_r$ **occurring in the binomial theorem are known as binomial coefficients.**

  - **There are** $(n + 1)$ **terms in the expansion of** $(a + b)^n$**.**

  - **In the successive terms of the expansion, the index of** a **goes on decreasing by unity starting from** n**, whereas the index of** b **goes on increasing by unity starting from 0.**

  - **In the expansion of** $(a + b)^n$**, the sum of indices of** a **and** b **in every term is** n**.**

  - **Special cases of expansion can be obtained by taking different values of** a **and** b**.**

    - **Taking** $a = x$ **and** $b = -y$**:**
      $(x - y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}y^2 - \ldots\ldots + (-1)^n\, {}^nC_ny^n$

    - **Taking** $a = 1$ **and** $b = x$**:**
      $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n$

    - **Taking** $a = 1$ **and** $b = 1$**:**
      $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_{n-1} + {}^nC_n$

    - **Taking** $a = 1$ **and** $b = -x$**:**
      $(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - \ldots\ldots + (-1)^n\, {}^nC_n\, x^n$

    - **Taking** $a = 1$ **and** $b = -1$**:**
      $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots\ldots + (-1)^n\, {}^nC_n$

Solved Examples

Example 1: Write the expansion of the expression $\left(1-\frac{3}{x}\right)^8$, where $x \neq 0$.

Solution:

Using Binomial theorem, we have

$$\left(1-\frac{3}{x}\right)^8 = {}^8C_0(1)^8 - {}^8C_1(1)^7\left(\frac{3}{x}\right) + {}^8C_2(1)^6\left(\frac{3}{x}\right)^2 - {}^8C_3(1)^5\left(\frac{3}{x}\right)^3 + {}^8C_4(1)^4\left(\frac{3}{x}\right)^4$$
$$- {}^8C_5(1)^3\left(\frac{3}{x}\right)^5 + {}^8C_6(1)^2\left(\frac{3}{x}\right)^6 - {}^8C_7(1)\left(\frac{3}{x}\right)^7 + {}^8C_8\left(\frac{3}{x}\right)^8$$
$$= 1 - 8\left(\frac{3}{x}\right) + 28\left(\frac{3}{x}\right)^2 - 56\left(\frac{3}{x}\right)^3 + 70\left(\frac{3}{x}\right)^4 - 56\left(\frac{3}{x}\right)^5 + 28\left(\frac{3}{x}\right)^6 - 8\left(\frac{3}{x}\right)^7 + \left(\frac{3}{x}\right)^8$$

Example 2: Find the value of $(202)^4$.

Solution:

We can write 202 as 200 + 2.

$\therefore (202)^4 = (200 + 2)^4$

On applying binomial theorem, we obtain

$(202)^4 = (200 + 2)^4$

$= {}^4C_0 (200)^4 + {}^4C_1 (200)^3(2) + {}^4C_2 (200)^2(2)^2 + {}^4C_3 (200)(2)^3 + {}^4C_4 (2)^4$

$= (200)^4 + 4 (200)^3(2) + 6 (200)^2(2)^2 + 4 (200)(2)^3 + (2)^4$

= 1600000000 + 64000000 + 960000 + 6400 + 16

= 1664966416

Example 3: Evaluate: $\left(1+\frac{x}{2}\right)^5 + \left(1-\frac{x}{2}\right)^5$.

Solution:

On using binomial theorem, we obtain

$$\left(1+\frac{x}{2}\right)^5 = {}^5C_0(1)^5 + {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 + {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 + {}^5C_5\left(\frac{x}{2}\right)^5$$

$$\left(1-\frac{x}{2}\right)^5 = {}^5C_0(1)^5 - {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 - {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$

Thus,

$$\left(1+\frac{x}{2}\right)^5 + \left(1-\frac{x}{2}\right)^5$$

$$= {}^5C_0(1)^5 + {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 + {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 + {}^5C_5\left(\frac{x}{2}\right)^5$$

$$+ {}^5C_0(1)^5 - {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 - {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$

$$= 2\left[{}^5C_0(1)^5 + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 + {}^5C_4(1)\left(\frac{x}{2}\right)^4\right]$$

$$= 2\left[1 + 10\left(\frac{x}{2}\right)^2 + 5\left(\frac{x}{2}\right)^4\right]$$

$$= 2 + 5x^2 + \frac{5}{8}x^4$$

General and Middle Term of A Binomial Expansion

- The $(r + 1)^{th}$ term or the **general term** of a binomial expansion is given by
  $\mathbf{T_{r+1} = {}^nC_r\, a^{n-r} b^r}$
- For example: The 15^{th} term in the expansion of $(5a + 3)^{25}$ is given by
  $T_{14+1} = {}^{25}C_{14}\, a^{25-14} b^{14} = {}^{25}C_{14}\, a^{11} b^{14}$
- To find the **middle term** of the expansion of $(a + b)^n$, the following formula is used:
- If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even, then $(n + 1)$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$, i.e., the $\left(\frac{n}{2}+1\right)^{th}$ term.
- If n is odd, then $n + 1$ is even. Hence, there will be two middle terms in the expansion, namely the $\left(\frac{n+1}{2}\right)^{th}$ term and the $\left(\frac{n+1}{2}+1\right)^{th}$ term.

- In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e., the $(n+1)^{th}$ term, as $2n$ is even.

Example 1: Find the term independent of p in the expansion of $\left(2p-\frac{1}{p}\right)^{16}$.

Solution:

We know that the general term i.e., the $(r+1)^{th}$ term of the binomial expansion of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r \; a^{n-r} b^r$

Hence,

$$T_{r+1} = {}^{16}C_r (2p)^{16-r}\left(-\frac{1}{p}\right)^r = (-1)^r \, {}^{16}C_r (2)^{16-r}(p)^{16-r}\left(\frac{1}{p}\right)^r = (-1)^r \, {}^{16}C_r (2)^{16-r}(p)^{16-2r}$$

The term will be independent of p, if the index of p is zero i.e., $16 - 2r = 0$.

This gives $r = 8$.

Hence, the 9^{th} term is independent of p and it is given by $(-1)^8 \, {}^{16}C_8 (2)^{16-8}(p)^{16-2\times 8} = \frac{16!}{8!8!}(2)^8(p)^0 = 12870\times(2)^8$

Example 2: In the expansion of $(2p+n)^7$, where n is an integer, the third and fourth terms are $6048\,p^5$ and $15120\,p^4$ respectively. Find the value of n.

Solution:

We know that the general term i.e., the $(r+1)^{th}$ term of the binomial expansion of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r \, a^{n-r} b^r$

Thus,

Third term = T_{2+1}

$= {}^7C_2 \, (2p)^{7-2} n^2$

$= 21 \times (2)^5 \, p^5 \, n^2$

The third term is given as 6048 p^5. Therefore,

$21 \times (2)^5 \, p^5 \, n^2 = 6048 \, p^5$

$\Rightarrow n^2 = 9$ ... (1)

Fourth term, $T_{3+1} = {}^7C_3(2p)^{7-3}n^3$

$= 35 \times (2)^4 \, p^4 \, n^3$

The fourth term is given as 15120 p^4. Therefore,

$35 \times (2)^4 \, p^4 \, n^3 = 15120 \, p^4$

$\Rightarrow n^3 = 27$ ... (2)

On dividing equation (2) by equation (1), we obtain

$n = \dfrac{27}{9} = 3$

Thus, the value of n is 3.

Example 3: Find the coefficient of x^2 in the expansion of $\left(\frac{1}{2} - \sqrt{x}\right)^{10}$.

Solution:

Suppose x^2 occurs in the $(r + 1)^{th}$ term of the expansion of $\left(\frac{1}{2} - \sqrt{x}\right)^{10}$.

Now, $T_{r+1} = {}^nC_r \, a^{n-r} b^r$

$\therefore T_{r+1} = {}^{10}C_r \left(\frac{1}{2}\right)^{10-r} \left(-\sqrt{x}\right)^r = {}^{10}C_r \left(\frac{1}{2}\right)^{10-r} (-1)^r \left(\sqrt{x}\right)^r$

Comparing the indices of x in x^2 and T_{r+1}, we obtain $r = 4$.

Thus, the coefficient of x^2 in the expansion of $\left(\frac{1}{2}-\sqrt{x}\right)^{10}$ is given by

T4+1 = ${}^{10}C_4\left(\frac{1}{2}\right)^{10-4}(-1)^4 = {}^{10}C_4\left(\frac{1}{2}\right)^6 = \frac{105}{32}$